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N 94-28680DOMAIN DECOMPOSITION METHODS FOR
NONCONFORMING FINITE ELEMENT SPACES OF LAGRANGE-TYPE*Lawrence C. Cowsar
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SUMMARY

In this article, we consider the application of three popular domain decomposition methods to Lagrange-type nonconforming finite element discretizations of scalar, self-adjoint, second order elliptic equations. The additive Schwarz method of Dryja and Widlund, the vertex space method of Smith, and the balancing method of Mandel applied to nonconforming elements are shown to converge at a rate no worse than their applications to the standard conforming piecewise linear Galerkin discretization. Essentially, the theory for the nonconforming elements is inherited from the existing theory for the conforming elements with only modest modification by constructing an isomorphism between the nonconforming finite element space and a space of continuous piecewise linear functions.

INTRODUCTION

We consider the convergence properties of domain decomposition methods applied to Lagrange-type nonconforming finite element discretizations of scalar, self-adjoint, second order elliptic problems. An isomorphism between the nonconforming finite element space with the natural norm induced by the elliptic problem and a conforming piecewise linear space with the H^1 -seminorm is constructed. Using the isomorphism, we are able to apply the existing analysis of domain decomposition methods for conforming elements to nonconforming elements with only modest modifications. As examples of this technique, we show that the operators arising in three popular domain decomposition methods, specifically, the additive Schwarz method of Dryja and Widlund [1], the vertex space method of Smith [2], and balancing method of Mandel [3], applied to nonconforming finite elements have condition numbers that satisfy the same bounds as the ones given in [4] and [5] for conforming finite elements.

The same technique was used in [6] and [7] to analyze the rate of convergence of balancing domain decomposition and the standard additive Schwarz method for the dual-variable mixed finite element formulation. Moreover, as a corollary of the analysis of Smith's method for the nonconforming spaces presented in this paper, we have a new bound for Smith's method applied to mixed finite elements.

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After the research for this paper was completed, the author was made aware of some related work done concurrently by Sarkis [8]. In particular, the isomorphism used herein was independently suggested by Sarkis for linear nonconforming elements. In [8], Sarkis constructs and analyzes special coarse spaces such that when the overlapping additive Schwarz method is applied, the condition number of the resulting operator is bounded by a constant times $(1 + \log(H/h))(1 + H/\delta)$ in both two and three dimensions. Here H and h are the characteristic sizes of the subdomains and mesh, respectively, and δ is a measure of the overlap of subdomains. The notable characteristic of Sarkis' bound is that the constant is independent of jumps in the coefficients across subdomain boundaries. If the techniques of this paper were used to derive bounds that were independent of the jumps in coefficients, the resulting bound would include one log factor in two dimensions using [1, 9], but two logs in three dimensions using [5, 10, 11].

The remainder of this paper is divided into six sections. In the next section, we set some notation, formulate the nonconforming problem, and construct an equivalent representation in terms of the nodal values. In Section 3, we construct an isomorphism between the nonconforming space and a continuous space of piecewise linear functions. The isomorphism is used in Section 4 to analyze the rate of convergence of the Dryja-Widlund additive Schwarz method. In the last three sections, we consider the substructuring methods of Smith and Mandel applied to the nonconforming problem.

PRELIMINARIES

We consider the following self-adjoint, uniformly elliptic problem for p on the polygonal domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, with boundary $\partial\Omega$:

$$-\nabla \cdot A \nabla p = f \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \partial\Omega, \quad (1)$$

where A is a uniformly positive definite, bounded, symmetric second order tensor, and $f \in L^2(\Omega)$. The uniform ellipticity of (1) implies the existence of positive constants c_* , c^* such that the following bound holds:

$$c_* \xi^T \xi \leq \xi^T A(x) \xi \leq c^* \xi^T \xi \quad \forall \xi \in \mathbb{R}^n, \forall x \in \Omega. \quad (2)$$

In order to set a length scale, we assume that the diameter of Ω is one. We introduce a two level quasi-regular triangulation of Ω : a division first into subdomains $\{\Omega_i\}_{i=1}^M$ with diameter $O(H)$, and a refinement of the first into elements with diameter $O(h)$. Following [12], define the scaled Sobolev norms

$$\|u\|_{1,\Omega_i}^2 = |u|_{1,\Omega_i}^2 + \frac{1}{H^2} \|u\|_{0,\Omega_i}^2, \quad \|u\|_{1/2,\partial\Omega_i}^2 = |u|_{1/2,\partial\Omega_i}^2 + \frac{1}{H} \|u\|_{0,\partial\Omega_i}^2,$$

where

$$\begin{aligned} \|u\|_{0,\Omega_i}^2 &= \int_{\Omega_i} |u(x)|^2 dx, & \|u\|_{0,\partial\Omega_i}^2 &= \int_{\partial\Omega_i} |u(s)|^2 ds, \\ |u|_{1,\Omega_i}^2 &= \int_{\Omega_i} |\nabla u(x)|^2 dx, & |u|_{1/2,\partial\Omega_i}^2 &= \int_{\partial\Omega_i} \int_{\partial\Omega_i} \frac{|u(t) - u(s)|^2}{|t - s|^n} dt ds. \end{aligned}$$

Let $\mathcal{N}(\Omega)$ be a finite dimensional nonconforming finite element space of Lagrange-type defined subordinate to the triangulation \mathcal{T} that vanishes at all degrees of freedom on $\partial\Omega$. Since $\mathcal{N}(\Omega)$ is of Lagrange-type, the elements in $\mathcal{N}(\Omega)$ may be expressed in terms of a nodal basis, and we may

identify an element in $\mathcal{N}(\Omega)$ with the values it attains at the nodal points. For convenience, we assume that the subdomains and the elements are triangular in two dimensions or tetrahedral in three dimensions. Extensions to other shape regular decompositions are straightforward.

We consider the problem of finding $p_h \in \mathcal{N}(\Omega)$ such that

$$d(p_h, q_h) = \int_{\Omega} f q_h dx \quad \forall q_h \in \mathcal{N}(\Omega), \quad (3)$$

where d is the generalized Dirichlet form:

$$d(p_h, q_h) = d_{\Omega}(p_h, q_h), \quad d_{\Omega'}(p_h, q_h) \equiv \sum_{\tau \in \mathcal{T}, \tau \subset \Omega'} \int_{\tau} A \nabla p_h \cdot \nabla q_h dx.$$

We now introduce several conventions used in this paper. In this paper, we shall only be concerned with the solution of this finite dimensional problem, and will henceforth drop the “h” subscript.

Having defined a parent finite element space of functions $\mathcal{X}(\Omega)$ with a nodal basis and a set $\Omega' \subset \Omega$, we will simply write $\mathcal{X}(\Omega')$ for the restriction of $\mathcal{X}(\Omega)$ to Ω' , i.e.

$$\mathcal{X}(\Omega') = \{\phi|_{\Omega'} \mid \phi \in \mathcal{X}(\Omega)\}.$$

By an abuse of notation, we consider an element $\phi \in \mathcal{X}(\Omega')$ also to be an element of $\mathcal{X}(\Omega)$ by setting ϕ to zero at all nodes outside of Ω' .

We will write $\mathcal{Q}_1 \simeq \mathcal{Q}_2$ if two quadratic forms \mathcal{Q}_1 and \mathcal{Q}_2 with the same domain \mathcal{D} are *equivalent*, i.e. if there exists constants $c_1, c_2 > 0$ such that

$$c_1 \mathcal{Q}_1(\phi, \phi) \leq \mathcal{Q}_2(\phi, \phi) \leq c_2 \mathcal{Q}_1(\phi, \phi), \quad \forall \phi \in \mathcal{D}.$$

In what follows, C will be used to denote a generic constant that may not be the same from one line to the next. This constant, as well as the constants involved in the equivalence of quadratic forms, will always be independent of h and H , but can depend on the constants in (2), the shape regularity of the subdomains, the degree of the nonconforming finite elements, and the regularity of the triangulation.

To conclude this section, we prove a lemma that provides an equivalent quadratic form for $d(\cdot, \cdot)$ in terms of the nodal degrees of freedom. The proof of this lemma was suggested by Joseph Pasciak in the context of the mixed finite methods considered in [6, 7].

Lemma 1 *Let $\Omega' \subseteq \Omega$ be the union of elements of \mathcal{T} . And let $A(x) = \alpha(x)\hat{A}(x)$, where α is a positive, piecewise constant function with value α_{τ} on $\tau \in \mathcal{T}$. Then for every $p \in \mathcal{N}(\Omega')$,*

$$d_{\Omega'}(p, p) \simeq \sum_{\substack{\tau \in \mathcal{T}, \\ \tau \subset \Omega'}} \alpha_{\tau} |\tau|^{1-2/n} \sum_{\substack{\text{nodes :} \\ n_i, n_j \in \tau}} (p(n_i) - p(n_j))^2. \quad (4)$$

The constants that appear in the definition of the equivalence do not depend on the constants in (2), but rather on constants that arise when A is replaced by \hat{A} .

Proof. The local kernel of $d_{\tau}(\cdot, \cdot)$ in $\mathcal{N}(\tau)$ is exactly the constant functions on τ since for $p \in \mathcal{N}(\tau)$

$$d_{\tau}(p, q) = 0 \quad \forall q \in \mathcal{N}(\tau), \quad \Longleftrightarrow \quad \nabla p = 0.$$

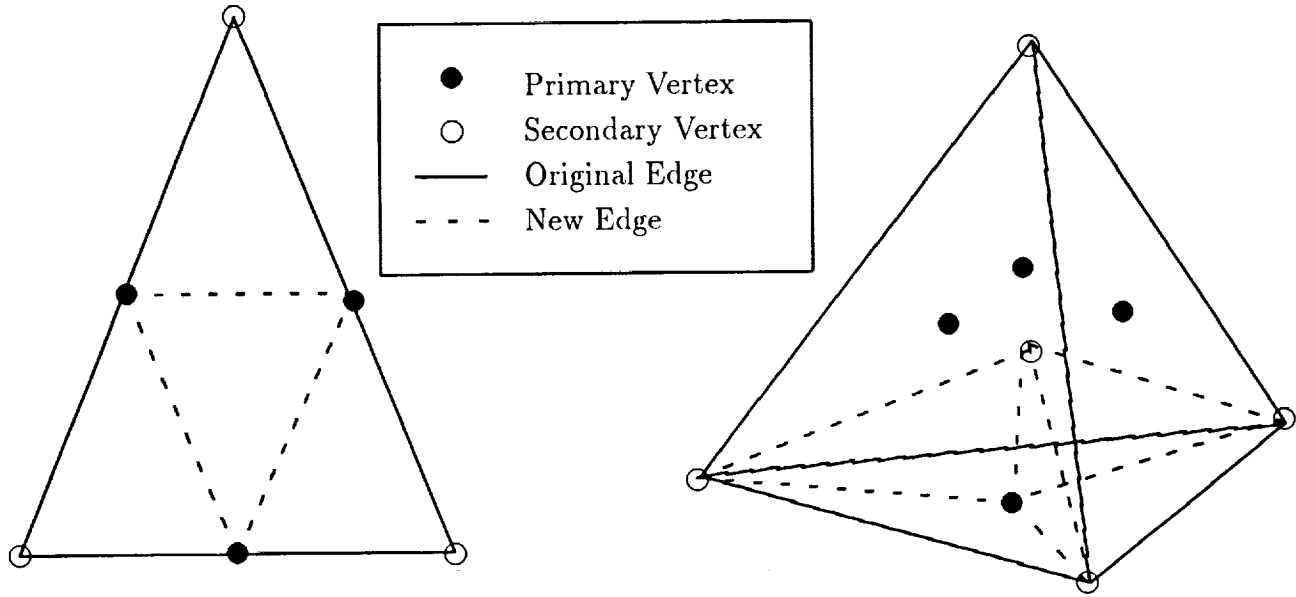


Figure 1: Refinement of the 2D P-1 element and a partial refinement of the 3D P-1 element.

Hence, $(d_\tau(\cdot, \cdot))^{1/2}$ is a norm on $\mathcal{N}(\tau)/\mathbb{R}$. Since all norms are equivalent on finite dimensional spaces, we see that

$$d_\tau(p, p) \simeq \alpha_\tau |\tau|^{1-2/n} \sum_{\substack{\text{nodes :} \\ n_i, n_j \in \tau}} (p(n_i) - p(n_j))^2,$$

by a simple scaling argument. The proof is completed by summing over the elements of \mathcal{T} in Ω' . \square

A CONFORMING EQUIVALENCE

In this section, we construct a conforming space that is isomorphic to $\mathcal{N}(\Omega)$ using the techniques in [6, 7] and recall some basic properties about the isomorphism.

Given an element $\tau \in \mathcal{T}$, let $\hat{\mathcal{T}}_\tau$ be a subtriangulation of τ such that the vertices of the subtriangulation include the vertices of τ and the nodal points in τ pertaining to the degrees of freedom of $\mathcal{N}(\tau)$. Every element in the new triangulation should have at least one vertex that corresponds to a nodal point of $\mathcal{N}(\tau)$. Moreover, the subtriangulations should be constructed in such a way that the union of subtriangulations gives rise to a refined quasi-regular triangulation of Ω which we denote by

$$\hat{\mathcal{T}} \equiv \bigcup_{\tau \in \mathcal{T}} \hat{\mathcal{T}}_\tau.$$

A vertex of $\hat{\mathcal{T}}$ will be called *primary* if it was a nodal point corresponding to a degree of freedom of $\mathcal{N}(\Omega)$; otherwise, we call the vertex *secondary*. We say that two vertices of the triangulation $\hat{\mathcal{T}}$ are *adjacent* if there exists an edge of $\hat{\mathcal{T}}$ connecting the vertices. An example of the subtriangulation of the P-1 element that has nodal degrees of freedom at the center of its edges (faces) is given in Figure 1.

Let $U_h(\Omega)$ denote the space of continuous piecewise linear functions subordinate to the triangulation \hat{T} that vanish on $\partial\Omega$. For $\Omega' \subset \Omega$, a union of elements, define $U_h(\Omega')$ by restriction, i.e.

$$U_h(\Omega') = \{u|_{\Omega'} \mid u \in U_h(\Omega)\}.$$

Since the functions in $U_h(\Omega')$ are naturally parameterized by the values they attain at the vertices, we can define a pseudo-interpolation operator $\tilde{T}^{\Omega'}$ into $U_h(\Omega')$ for any function ϕ defined at the primary vertices contained in Ω' by

$$\tilde{T}^{\Omega'}\phi(x) = \begin{cases} 0, & \text{if } x \in \partial\Omega' \cap \partial\Omega; \\ \phi(x), & \text{if } x \text{ is a primary vertex not in } \partial\Omega' \cap \partial\Omega; \\ \text{The average of all adjacent primary vertices on the boundary} \\ \text{of } \Omega', & \text{if } x \text{ is a secondary vertex in } \partial\Omega' \setminus \partial\Omega; \\ \text{The average of all adjacent primary vertices, if } x \text{ is a secondary} \\ \text{vertex in the interior of } \Omega'; \\ \text{The continuous piecewise linear interpolant of the above vertex} \\ \text{values, if } x \text{ is not a vertex of } \hat{T}. \end{cases} \quad (5)$$

Since $\tilde{T}^{\Omega'}$ is well defined for any function defined at the primary vertices, by an abuse of notation, we can understand $\tilde{T}^{\Omega'}$ both as a map from $\mathcal{N}(\Omega')$ into $U_h(\Omega')$ and a map from $U_h(\Omega')$ into itself.

For any Ω' that is the union of elements in \mathcal{T} , let $\tilde{U}_h(\Omega') \subset U_h(\Omega')$ denote the range of $\tilde{T}^{\Omega'}$; that is,

$$\tilde{U}_h(\Omega') = \{\psi = \tilde{T}^{\Omega'}q, q \in \mathcal{N}(\Omega')\}.$$

We now prove that $\tilde{T}^{\Omega'} : \mathcal{N}(\Omega') \rightarrow \tilde{U}_h(\Omega')$ preserves the norm induced by the bilinear form $d_{\Omega'}(\cdot, \cdot)$ on $\mathcal{N}(\Omega')$ and the H^1 -seminorm on $\tilde{U}_h(\Omega')$. Since $\tilde{T}^{\Omega'}$ is a bijection between $\mathcal{N}(\Omega')$ and $\tilde{U}_h(\Omega')$ by construction, this proves that $\mathcal{N}(\Omega')$ and $\tilde{U}_h(\Omega')$ are isomorphic.

Theorem 2 *Let $\Omega' \subset \Omega$ be the union of elements. Then for all $p \in \mathcal{N}(\Omega')$,*

$$d_{\Omega'}(p, p) \simeq |\tilde{T}^{\Omega'}p|_{1, \Omega'}^2. \quad (6)$$

Proof. This proof is an expanded version of the proof given in [7]. Recall that for $\phi \in U_h(\Omega')$,

$$|\phi|_{1, \Omega'}^2 \simeq \sum_{\substack{\tau \in \hat{T}, \\ \tau \subset \Omega'}} |\tau|^{1-2/n} \sum_{\substack{\text{vertices :} \\ v_i, v_j \in \tau}} (\phi(v_i) - \phi(v_j))^2. \quad (7)$$

By virtue of Lemma 1 and Equation (7), it is enough to show that

$$\sum_{\substack{\tau \in \mathcal{T}, \\ \tau \subset \Omega'}} |\tau|^{1-2/n} \sum_{\substack{\text{nodes :} \\ n_i, n_j \in \tau}} (p(n_i) - p(n_j))^2 \simeq \sum_{\substack{\tau \in \hat{T}, \\ \tau \subset \Omega'}} |\tau|^{1-2/n} \sum_{\substack{\text{vertices :} \\ v_i, v_j \in \tau}} ((\tilde{T}^{\Omega'}p)(v_i) - (\tilde{T}^{\Omega'}p)(v_j))^2. \quad (8)$$

Since vertices of \hat{T}_τ contain the nodal points of τ and $p = \tilde{T}^{\Omega'}p$ at these points, we have

$$\sum_{\substack{\text{nodes :} \\ n_i, n_j \in \tau}} (p(n_i) - p(n_j))^2 \leq C \sum_{\hat{\tau} \in \hat{T}_\tau} \sum_{\substack{\text{vertices :} \\ v_i, v_j \in \hat{\tau}}} ((\tilde{T}^{\Omega'}p)(v_i) - (\tilde{T}^{\Omega'}p)(v_j))^2,$$

where the constant is controlled by the regularity of the subtriangulation. Hence, by summing over the elements of \mathcal{T} in Ω' , we conclude that the right hand side of (8) dominates the left hand side.

To prove that the left hand side dominates the right, we note that the differences in the right hand side are of three types: the difference at two primary vertices, the difference at two secondary vertices, and the difference at a primary and a secondary vertex. Since p and $\tilde{\mathcal{I}}^{\Omega'} p$ agree at primary vertices of $\hat{\mathcal{T}}$, the difference at two primary vertices occurs as a term in the left hand side. For two secondary vertices v_1, v_2 in an element $\tau \in \hat{\mathcal{T}}$ containing a primary vertex v_p , we see that

$$\left((\tilde{\mathcal{I}}^{\Omega'} p)(v_1) - (\tilde{\mathcal{I}}^{\Omega'} p)(v_2) \right)^2 \leq 2 \left((\tilde{\mathcal{I}}^{\Omega'} p)(v_1) - (\tilde{\mathcal{I}}^{\Omega'} p)(v_p) \right)^2 + 2 \left((\tilde{\mathcal{I}}^{\Omega'} p)(v_2) - (\tilde{\mathcal{I}}^{\Omega'} p)(v_p) \right)^2.$$

Hence, it is enough to bound the difference at a secondary and primary vertex by terms in the left hand side of (8).

Let v_{n+1} be a secondary vertex with adjacent primary vertices v_1, \dots, v_n , and let $p_j = p(v_j)$. Noting that for $j = 1, \dots, n$

$$(\tilde{\mathcal{I}}^{\Omega'} p)(v_j) = p_j, \quad (\tilde{\mathcal{I}}^{\Omega'} p)(v_{n+1}) = \frac{1}{n} \sum_{j=1}^n (\tilde{\mathcal{I}}^{\Omega'} p)(v_j) = \frac{1}{n} \sum_{j=1}^n p_j,$$

we see that

$$\left((\tilde{\mathcal{I}}^{\Omega'} p)(v_{n+1}) - (\tilde{\mathcal{I}}^{\Omega'} p)(v_i) \right)^2 = \frac{1}{n^2} \left(\sum_{j=1}^n (p_j - p_i) \right)^2 \leq \frac{n}{n^2} \sum_{j=1}^n (p_j - p_i)^2,$$

by the Cauchy-Schwarz inequality. The proof is completed by summing over all triangles of $\hat{\mathcal{T}}$. The number of such terms, and hence the constant in the bound, is controlled since the regularity of the mesh implies that there is an a priori maximum number of adjacent elements that can share a secondary point. \square

Using the techniques in the proof of Theorem 2, the following lemma is easy to prove.

Lemma 3 *There exists a constant C depending only on the regularity of the triangulation \mathcal{T} and the degree of the nonconforming space such that for any $\Omega' \subset \Omega$, the union of elements of \mathcal{T} ,*

$$|\tilde{\mathcal{I}}^{\Omega'} \phi|_{k, \Omega'} \leq C |\phi|_{k, \Omega'} \quad \forall \phi \in U_h(\Omega'), \quad k = 0, 1. \quad (9)$$

THE DRYJA-WIDLUND ADDITIVE SCHWARZ METHOD

The presentation in this section and the next follows the treatment of Schwarz methods given by Dryja and Widlund in [4]. We concentrate only on the additive Schwarz methods with exact solves. The convergence rate of the multiplicative Schwarz method may be estimated in terms of the same quantities (see [13]) and is easily worked out. Extensions to inexact solves are likewise direct.

Recall that the additive Schwarz method with exact solves for (3) is completely determined by a decomposition of the finite element space $\mathcal{N}(\Omega) = \mathcal{N}_0 + \mathcal{N}_1 + \dots + \mathcal{N}_M$. For each subspace \mathcal{N}_i , define an operator $P_i : \mathcal{N}(\Omega) \rightarrow \mathcal{N}_i$ by

$$d(P_i p, q) = d(p, q) \quad \forall q \in \mathcal{N}_i. \quad (10)$$

The *additive Schwarz algorithm* with exact solves for (3) involves the solution of

$$Pp = \hat{f}, \quad P \equiv \sum_{i=0}^M P_i, \quad \hat{f} \equiv \sum_{i=0}^M f_i, \quad (11)$$

where $f_i \in \mathcal{N}_i$ is defined by

$$d(f_i, q) = \int_{\Omega} f_i q \, dx \quad \forall q \in \mathcal{N}_i.$$

Abstract bounds on the condition number of P have been derived in terms of two quantities, C_0 and the spectral radius of \mathcal{E} , which we now define. Let C_0 be a constant such that for every $p \in \mathcal{N}$ there exists a representation $p = \sum_{i=0}^M p_i$ with $p_i \in \mathcal{N}_i$ satisfying

$$\sum_{i=0}^M d(p_i, p_i) \leq C_0 d(p, p). \quad (12)$$

Let $\rho(\mathcal{E})$ denote the spectral radius of $\mathcal{E} = \{\epsilon_{ij}\}$, the matrix of strengthened Cauchy-Schwarz constants; that is, ϵ_{ij} is the smallest constant for which

$$|d(p_i, p_j)| \leq \epsilon_{ij} d(p_i, p_i)^{\frac{1}{2}} d(p_j, p_j)^{\frac{1}{2}} \quad \forall p_i \in \mathcal{N}_i, \forall p_j \in \mathcal{N}_j, i, j \geq 1. \quad (13)$$

The next theorem, due to Dryja and Widlund [14], bounds the condition number of the additive Schwarz method in terms of C_0 and $\rho(\mathcal{E})$:

Theorem 4 *The eigenvalues and the condition number $\kappa(P)$ of P satisfy*

$$\lambda_{\min}(P) \geq C_0^{-1}, \quad \lambda_{\max}(P) \leq (\rho(\mathcal{E}) + 1), \quad \kappa(P) \leq C_0(\rho(\mathcal{E}) + 1). \quad (14)$$

To construct the decomposition of $\mathcal{N}(\Omega)$ to be used in our application of the additive Schwarz algorithm for nonconforming elements, we first create an overlapping decomposition of the domain Ω by extending each subdomain Ω_i to a larger region Ω'_i which is also the union of elements of \mathcal{T} . We characterize the extent of the overlap of the partition $\{\Omega'_i\}_{i=1}^M$ by δ , where

$$\delta = \min_{i=1, \dots, M} \text{dist}(\partial\Omega_i \setminus \partial\Omega, \partial\Omega'_i \setminus \partial\Omega).$$

The decomposition $\{\Omega'_i\}_{i=1}^M$ gives rise to a natural decomposition of $\mathcal{N}(\Omega)$ by letting $\mathcal{N}_i \subset \mathcal{N}(\Omega)$ denote the set of functions that vanish at all nodes in the closure of $(\Omega \setminus \Omega'_i)$. In order to provide a mechanism for global exchange of information between subdomains so as to enhance the rate of convergence, we also use a low dimensional space defined by

$$\mathcal{N}_0 = \{p \in \mathcal{N}(\Omega) \mid p = \mathcal{I}^{\mathcal{N}} \phi, \phi \in U_H(\Omega)\},$$

where $\mathcal{I}^{\mathcal{N}}$ is nodal interpolation into $\mathcal{N}(\Omega)$, and $U_H(\Omega)$ is the space of continuous functions that are linear on each subdomain Ω_i . Note that the subspaces for the nonconforming space are exactly the nodal interpolants of the standard decomposition of the conforming space $U_h(\Omega)$, namely, $U_h(\Omega) \cap H_0^1(\Omega'_i)$.

In the following lemma we recall the crux of the proof due to Dryja and Widlund (Theorem 3 of [4]) that the Schwarz method applied to the conforming Galerkin discretization has a condition number that is $O(1 + (H/\delta))$.

Lemma 5 For every $\phi \in U_h(\Omega)$, there exists a decomposition $\phi = \sum_{i=0}^M \phi_i$ with $\phi_0 \in U_H(\Omega)$, $\phi_i \in U_h(\Omega) \cap H_0^1(\Omega'_i)$, $1 \leq i \leq M$ and a constant C independent of h , H , and δ , such that

$$\sum_{i=0}^M \|\phi_i\|_{1,\Omega}^2 \leq C \left(1 + \frac{H}{\delta}\right) \|\phi\|_{1,\Omega}^2. \quad (15)$$

We now show that the application of the Schwarz method to the nonconforming space converges at the same rate.

Theorem 6 The condition number $\kappa(P)$ of the additive Schwarz operator P defined by (11) induced by the decomposition $\mathcal{N}(\Omega) = \mathcal{N}_0 + \dots + \mathcal{N}_M$ of the nonconforming finite element space satisfies

$$\kappa(P) \leq C \left(1 + \frac{H}{\delta}\right).$$

The constant C is independent of h , δ , and H .

Proof. The verification that the largest eigenvalue of P is bounded by a constant is standard. Since $d(p_i, p_j) \equiv 0$ for $p_i \in \mathcal{N}_i$, $p_j \in \mathcal{N}_j$ with $\Omega'_i \cap \Omega'_j = \emptyset$, P may be written as the sum of an a priori bounded number of disjoint projections. Since projections have unit norm, a constant bound on the largest eigenvalue of P is immediate. See, e.g., Lemma 3.1 of [2].

For $p \in \mathcal{N}(\Omega)$, let $(\tilde{\mathcal{T}}^\Omega p)_i$ denote the decomposition of $\tilde{\mathcal{T}}^\Omega p \in U_h(\Omega)$ arising in Lemma 5, and set $p_i = \mathcal{I}^\mathcal{N}((\tilde{\mathcal{T}}^\Omega p)_i)$. It is easy to check that $p_i \in \mathcal{N}_i$ and $p = \sum_{i=0}^M p_i$. Using Theorem 2 and Lemma 3, we see that for $i = 0, \dots, M$,

$$d(p_i, p_i) \leq C \|\tilde{\mathcal{T}}^\Omega((\tilde{\mathcal{T}}^\Omega p)_i)\|_{1,\Omega}^2 \leq C \|(\tilde{\mathcal{T}}^\Omega p)_i\|_{1,\Omega}^2.$$

Summing and applying Lemma 5 and Theorem 2, we conclude that

$$\sum_{i=0}^M d(p_i, p_i) \leq C \sum_{i=0}^M \|(\tilde{\mathcal{T}}^\Omega p)_i\|_{1,\Omega}^2 \leq C \left(1 + \frac{H}{\delta}\right) \|\tilde{\mathcal{T}}^\Omega p\|_{1,\Omega}^2 \leq C \left(1 + \frac{H}{\delta}\right) d(p, p).$$

Hence, C_0 in (12) is bounded by $C(1 + H/\delta)$. An application of Theorem 4 completes the proof. \square

SUBSTRUCTURING DOMAIN DECOMPOSITION

The remaining two methods considered in this paper are domain decomposition methods applied to a reduced problem involving only the degrees of freedom on the internal interfaces of subdomains $\Gamma = \cup_{i=1}^M \partial\Omega_i \setminus \partial\Omega$. Following [4], we recall the construction of the reduced problem. Since $\mathcal{N}(\Omega)$ is of Lagrange-type, we may associate with functions $p, q \in \mathcal{N}(\Omega)$ the vectors of values they attain at the nodes. Let x and y denote the vectors of nodal values of p and q , respectively, and $x^{(i)}, y^{(i)}$ the subvectors of degrees of freedom in $\bar{\Omega}_i$. Let $D^{(i)}$ denote the local stiffness matrix arising from $d_{\Omega_i}(\cdot, \cdot)$, and let D denote the global stiffness matrix, i.e.

$$x^{(i)T} D^{(i)} y^{(i)} = d_{\Omega_i}(p, q), \quad x^T D y = \sum_{i=1, \dots, M} x^{(i)T} D^{(i)} y^{(i)} = d(p, q).$$

For each subdomain, we can partition the degrees of freedom $x^{(i)}$ into two sets, the ones related to nodes on the boundary of Ω_i denoted $x_B^{(i)}$, and the ones corresponding to nodes in the interior of Ω_i denoted $x_I^{(i)}$. Such a partitioning induces a partitioning of $D^{(i)}$ given by

$$x^{(i)T} D^{(i)} y^{(i)} = \begin{pmatrix} x_I^{(i)} \\ x_B^{(i)} \end{pmatrix}^T \begin{pmatrix} D_{II}^{(i)} & D_{IB}^{(i)} \\ D_{IB}^{(i)T} & D_{BB}^{(i)} \end{pmatrix} \begin{pmatrix} y_I^{(i)} \\ y_B^{(i)} \end{pmatrix}.$$

The interior unknowns of each subdomain may be eliminated in terms of the boundary unknowns. The resulting matrix, S , is the Schur complement with respect to the interface unknowns defined by

$$x_B^T S y_B = \sum_{i=1, \dots, M} x_B^{(i)T} S^{(i)} y_B^{(i)}, \quad \text{where} \quad S^{(i)} = D_{BB}^{(i)} - D_{IB}^{(i)T} (D_{II}^{(i)})^{-1} D_{IB}^{(i)}.$$

It will be convenient to work with the bilinear forms induced by S and $S^{(i)}$, and so we define

$$s(p, q) = x_B^T S y_B, \quad s_i(p, q) = x_B^{(i)T} S^{(i)} y_B^{(i)}.$$

For a function $p \in \mathcal{N}(\Omega)$, we note that unlike conforming spaces, the restriction of p to the interfaces, $p|_\Gamma$, is not solely determined by the nodal values on Γ since $\mathcal{N}(\Omega)$ is nonconforming. Hence, we are careful to understand $\mathcal{N}(\Gamma)$ as a subset of $\mathcal{N}(\Omega)$ parameterized by the nodal values on Γ consisting of the discrete harmonic extension of the nodal values to the interior of the subdomains. Specifically, if $p \in \mathcal{N}(\Gamma)$ has the vector of nodal values $x_B^{(i)}$ on $\partial\Omega_i$, then $p|_{\Omega_i}$ is the function associated with the vector of nodal values $(x_I^{(i)}, x_B^{(i)})^T$ where $x_I^{(i)}$ satisfies

$$D_{II}^{(i)} x_I^{(i)} = -D_{IB}^{(i)} x_B^{(i)}.$$

A linear functional g is easily constructed such that finding $p \in \mathcal{N}(\Gamma)$ satisfying

$$s(p, q) = g(q) \quad \forall q \in \mathcal{N}(\Gamma) \tag{16}$$

is essentially equivalent to (3).

We now construct a conforming space of functions that is isomorphic to $\mathcal{N}(\Gamma)$ with the norm induced by the bilinear form $s(\cdot, \cdot)$. Let $U_h(\Gamma)$ denote the restriction of $U_h(\Omega)$ to $\cup_{i=1}^M \partial\Omega_i$. Since functions in $U_h(\Gamma)$ vanish on $\partial\Omega$ (because functions in $U_h(\Omega)$ do), functions in $U_h(\Gamma)$ can be parameterized in the natural nodal basis by the values they attain at the vertices of $\hat{\mathcal{T}}$ in Γ . Analogous to (5), for Γ' the union of edges (and faces in 3D) in the triangulation \mathcal{T} and ϕ a function defined at the primary vertices in Γ' , define a pseudo-interpolant $\tilde{\mathcal{I}}^{\Gamma'} \phi \in U_h(\Gamma')$ by

$$\tilde{\mathcal{I}}^{\Gamma'} \phi(x) = \begin{cases} 0, & \text{if } x \in \Gamma' \cap \partial\Omega; \\ \phi(x), & \text{if } x \text{ is a primary vertex not in } \Gamma' \cap \partial\Omega; \\ \text{The average of all adjacent primary vertices on } \Gamma' \text{ if } x \text{ is a} \\ \quad \text{secondary vertex on } \Gamma'; \\ \text{The continuous piecewise linear interpolant of the above vertex} \\ \quad \text{values, if } x \text{ is not a vertex of } \hat{\mathcal{T}}. \end{cases} \tag{17}$$

Note that if $\Gamma' = \partial\Omega'$, then $\tilde{T}'\phi = (\tilde{T}'\tilde{\phi})|_{\partial\Omega'}$ for all $\tilde{\phi}$ in $\mathcal{N}(\Omega')$ that agree with ϕ at the nodal degrees of freedom of $\partial\Omega'$.

Since \tilde{T}' is well defined for any function defined at primary vertices, by an abuse of notation, we can understand \tilde{T}' both as a map from $\mathcal{N}(\Gamma')$ into $U_h(\Gamma')$ and a map from $U_h(\Gamma')$ into $U_h(\Gamma')$. We denote the range of \tilde{T}' by

$$\tilde{U}_h(\Gamma') = \{(\tilde{T}'\psi)|_{\Gamma'} | \psi \in U_h(\Gamma')\}.$$

The equivalences in the following lemma are a combination of the standard trace theorem and an extension theorem for $\tilde{U}_h(\partial\Omega_i)$. In particular, the proof of this lemma given in [6] shows that the space $\tilde{U}_h(\Omega_i)$ is rich enough to inherit the Extension Theorem of Widlund [15] from $U_h(\Omega_i)$.

Lemma 7 For $\hat{\phi} \in \tilde{U}_h(\partial\Omega_i)$,

$$\|\hat{\phi}\|_{1/2, \partial\Omega_i} \simeq \inf_{\substack{\phi \in \tilde{U}_h(\Omega_i) \\ \phi|_{\partial\Omega_i} = \hat{\phi}}} \|\phi\|_{1, \Omega_i}, \quad |\hat{\phi}|_{1/2, \partial\Omega_i} \simeq \inf_{\substack{\phi \in \tilde{U}_h(\Omega_i) \\ \phi|_{\partial\Omega_i} = \hat{\phi}}} |\phi|_{1, \Omega_i}. \quad (18)$$

Additionally, there exists a constant C independent of mesh parameters such that

$$|\tilde{T}^{\partial\Omega_i}\hat{\phi}|_{k, \partial\Omega_i} \leq C|\hat{\phi}|_{k, \partial\Omega_i}, \quad \forall \hat{\phi} \in U_h(\partial\Omega_i), \quad k = 0, 1/2. \quad (19)$$

The following theorem plays the role of Theorem 2 for the interface problem.

Theorem 8 For all $p \in \mathcal{N}(\Gamma)$,

$$s_i(p, p) \simeq |\tilde{T}^{\partial\Omega_i}p|_{1/2, \partial\Omega_i}^2. \quad (20)$$

Proof. By a direct computation followed by an application of Theorem 2 and Lemma 7 noting that $\tilde{U}_h(\Omega_i) = \tilde{T}^{\Omega_i}(\mathcal{N}(\Omega_i))$, we see that

$$s_i(p, p) = \inf_{\substack{\tilde{p} \in \mathcal{N}(\Omega_i) \\ \tilde{p}|_{\partial\Omega_i} = p}} d_{\Omega_i}(\tilde{p}, \tilde{p}) \simeq \inf_{\substack{\tilde{p} \in \mathcal{N}(\Omega_i) \\ \tilde{p}|_{\partial\Omega_i} = p}} |\tilde{T}^{\Omega_i}\tilde{p}|_{1, \Omega_i}^2 \simeq |\tilde{T}^{\partial\Omega_i}p|_{1/2, \partial\Omega_i}^2.$$

□

SMITH'S VERTEX SPACE METHOD

Smith's vertex space method [2] is an additive Schwarz method applied to the interface problem (16). The decomposition of $\mathcal{N}(\Gamma)$ is constructed slightly differently in two and three dimensions. In both cases, we first partition Γ into overlapping subsets based on its decomposition as the boundary of subdomains. In two dimensions, for each vertex V_j of Γ , let $\Gamma_\delta^{V_j}$ denote the set of points on Γ that are less than a distance δ from V_j . For each edge E_i of Γ , let $\Gamma_\delta^{E_i}$ denote the interior of the edge E_i . In three dimensions, for each vertex V_j , each edge E_i , and each face F_k of Γ , define $\Gamma_\delta^{V_j}$ as above, let $\Gamma_\delta^{F_k}$ denote the interior of the face F_k , and let $\Gamma_\delta^{E_i}$ denote the set of all points in strips of width δ on all faces which share the common edge E_i .

Understanding the set of faces to be empty in two dimensions, the decomposition of Γ into subsets induces a decomposition of $\mathcal{N}(\Gamma)$ by considering

$$\mathcal{N}(\Gamma) = \sum_{G \in \{H, E_i, V_j, F_k\}} \mathcal{N}(\Gamma_\delta^G),$$

where for $G \in \{E_i, V_j, F_k\}$, $\mathcal{N}(\Gamma_\delta^G) \subset \mathcal{N}(\Gamma)$ are those functions that vanish at all nodal points on Γ that are outside of the set Γ_δ^G , and $\mathcal{N}(\Gamma_\delta^H) \subset \mathcal{N}(\Gamma)$ are those functions that are the nodal interpolant of the restriction to Γ of continuous functions that are linear on each subdomain Ω_i and vanish on $\partial\Omega$.

The following lemma is the crux of the analysis of Smith's method by Dryja and Widlund [4] for conforming elements.

Lemma 9 *For every $\phi \in U_h(\Gamma)$, there exists a decomposition*

$$\phi = \sum_{G \in \{H, E_i, V_j, F_k\}} \phi_G$$

with $\phi_H \in U_H(\Gamma)$, $\phi_G \in U_h(\Gamma_\delta^G) = U_h(\Gamma) \cap H_0^1(\Gamma_\delta^G)$ for $G \in \{E_i, V_j, F_k\}$ such that

$$\sum_{G \in \{H, E_i, V_j, F_k\}} \sum_{i=1}^M |\phi_G|_{1/2, \partial\Omega_i}^2 \leq C (1 + \log(H/\delta))^2 \sum_{i=1}^M |\phi|_{1/2, \partial\Omega_i}^2. \quad (21)$$

The constant C is independent of the choice of ϕ , and the mesh parameters h , H , and δ .

Let $P_\Gamma : \mathcal{N}(\Gamma) \rightarrow \mathcal{N}(\Gamma)$ denote the additive Schwarz operator defined by (10) with the bilinear form $d(\cdot, \cdot)$ replaced by the interface form $s(\cdot, \cdot)$ and the decomposition of $\mathcal{N}(\Omega)$ replaced by the decomposition of $\mathcal{N}(\Gamma)$ described above. We now prove that the condition number of \mathcal{N}_Γ for the nonconforming space has the same bound given in [4] for the similar operator for the conforming finite element space.

Theorem 10 *The condition number of the additive Schwarz operator P_Γ for Smith's decomposition for the nonconforming finite element discretization satisfies*

$$\kappa(P_\Gamma) \leq C((1 + \log(H/\delta))^2). \quad (22)$$

The constant C is independent of the mesh parameters h , H , and δ .

Proof. As in the proof of Theorem 6, P_Γ may be written as the sum of an a priori bounded number of disjoint projections, and so the largest eigenvalue of P_Γ is bounded by a constant.

To bound the smallest eigenvalue, we also proceed as in the proof of Theorem 6. For $p \in \mathcal{N}(\Gamma)$, set $p_G = \mathcal{I}_\Gamma^\mathcal{N}((\tilde{T}p)_G)$, $G \in \{H, E_i, V_j, F_k\}$, where $\mathcal{I}_\Gamma^\mathcal{N}$ is interpolation at the nodes on Γ into $\mathcal{N}(\Gamma)$ and $(\tilde{T}p)_G$ is the decomposition of $\tilde{T}p \in U_h(\Gamma)$ that arises in Lemma 9. Since $\tilde{T}p$ and p agree at the nodal degrees of freedom of $\mathcal{N}(\Gamma)$, and

$$\mathcal{N}(\Gamma_\delta^H) = \mathcal{I}_\Gamma^\mathcal{N}(U_H(\Gamma)), \quad \mathcal{N}(\Gamma_\delta^G) = \mathcal{I}_\Gamma^\mathcal{N}(U_h(\Gamma_\delta^G)) \quad \forall G \in \{E_i, V_j, F_k\},$$

it is easy to check that

$$p = \sum_{G \in \{H, E_i, V_j, F_k\}} p_G.$$

Working one subdomain at a time and using Theorem 8 and Lemma 3, we see that for $G = H$ and for $G \in \{E_i, V_j, F_k\}$ such that $\Gamma_\delta^G \cap \partial\Omega_i \neq \emptyset$ we have

$$s_i(p_G, p_G) \leq C |\tilde{T}^{\partial\Omega_i} p_G|_{1/2, \partial\Omega_i}^2 = C |\tilde{T}^{\partial\Omega_i}((\tilde{T}p)_G)|_{1/2, \partial\Omega_i}^2 \leq C |(\tilde{T}p)_G|_{1/2, \partial\Omega_i}^2. \quad (23)$$

Assume that we can prove that there exists a constant independent of h , H and δ such that

$$\sum_{i=1}^M |\tilde{\mathcal{T}}p|_{1/2, \partial\Omega_i}^2 \leq C \sum_{i=1}^M |\tilde{\mathcal{T}}^{\partial\Omega_i} p|_{1/2, \partial\Omega_i}^2 \quad \forall p \in \mathcal{N}(\Gamma). \quad (24)$$

Then by summing (23) over subdomains and subspaces, noting that $s_i(p_G, p_G) = 0$ if $\Gamma_\delta^G \cap \partial\Omega_i = \emptyset$, and applying Lemma 9, Equation (24), and Theorem 8, we see that

$$\begin{aligned} \sum_{G \in \{H, E_i, V_j, F_k\}} s(p_G, p_G) &= \sum_{G \in \{H, E_i, V_j, F_k\}} \sum_{i=1}^M s_i(p_G, p_G) \leq C (1 + \log(H/\delta))^2 \sum_{i=1}^M |\tilde{\mathcal{T}}p|_{1/2, \partial\Omega_i}^2 \\ &\leq C (1 + \log(H/\delta))^2 \sum_{i=1}^M |\tilde{\mathcal{T}}^{\partial\Omega_i} p|_{1/2, \partial\Omega_i}^2 \leq C (1 + \log(H/\delta))^2 s(p, p). \end{aligned}$$

The proof of the condition number bound now follows from an application of Theorem 4, and we are only left to verify (24).

Define a pseudo-interpolant $\tilde{\mathcal{T}}^{\Omega \setminus \Gamma} : \mathcal{N}(\Omega) \rightarrow U_h(\Omega)$ by (5), noting that the boundary of $\Omega \setminus \Gamma$ is $\partial\Omega \cup \Gamma$. Using the techniques in the proof of Theorem 2, it is easy to show that there exists a constant C_1 depending only on the regularity of the mesh and the degree of the nonconforming space such that

$$\sum_{i=1}^M |\tilde{\mathcal{T}}^{\Omega \setminus \Gamma} p|_{1, \Omega_i}^2 \leq C_1 \sum_{i=1}^M |\tilde{\mathcal{T}}^{\partial\Omega_i} p|_{1, \Omega_i}^2 \quad \forall p \in \mathcal{N}(\Omega).$$

By Lemma 7, for each $p \in \mathcal{N}(\Gamma)$ there exists an extension $p^E \in \mathcal{N}(\Omega)$ that agrees with p at the nodal points on Γ such that

$$|\tilde{\mathcal{T}}^{\Omega} p^E|_{1, \Omega_i}^2 \leq C |\tilde{\mathcal{T}}^{\partial\Omega_i} p|_{1/2, \partial\Omega_i}^2 \quad i = 1, \dots, M.$$

Combining these results after another application of Lemma 7 with $\hat{\phi} = \tilde{\mathcal{T}}p$, we conclude that

$$\sum_{i=1}^M |\tilde{\mathcal{T}}p|_{1/2, \partial\Omega_i}^2 \leq C \sum_{i=1}^M |\tilde{\mathcal{T}}^{\Omega \setminus \Gamma} p^E|_{1, \Omega_i}^2 \leq C \sum_{i=1}^M |\tilde{\mathcal{T}}^{\partial\Omega_i} p^E|_{1, \Omega_i}^2 \leq C |\tilde{\mathcal{T}}^{\partial\Omega} p|_{1/2, \partial\Omega}^2,$$

which verifies (24). \square

In [6], the interface form arising from the discretization by mixed finite elements of (1) was shown to satisfy Theorem 8 with $\mathcal{N}(\Gamma)$ replaced by the appropriate space of interelement multipliers. Hence, the proof given above is applicable to discretization by mixed finite elements, and we arrive at the following corollary.

Corollary 11 *The application of Smith's decomposition method to the dual-variable mixed finite element formulation discussed in [6] results in an operator whose condition number grows at worst like $O((1 + \log(H/\delta))^2)$.*

BALANCING DOMAIN DECOMPOSITION

As the final domain decomposition method considered in this paper, we investigate the balancing domain decomposition method of Mandel [3] applied to nonconforming finite elements. The method

involves the iterative solution (usually by conjugate gradients) of (16) preconditioned by the balancing preconditioner described in Algorithm 1 below. Each iteration involves the solution of a local problem with Dirichlet data, a local problem with Neumann data, and a “coarse-grid” problem to propagate information globally and to insure the consistency of the Neumann problem. The theory and practical performance of balancing domain decomposition for the standard conforming Galerkin finite element method and mixed finite element method are the subjects of [5] and [6], respectively. As in previous sections, we will deduce the convergence theory for the nonconforming spaces from the conforming theory in [5] using the isomorphism introduced in the fifth section of this paper.

One remarkable property of balancing domain decomposition is that the bound on the condition number of the preconditioned operator is independent of jumps in coefficients across subdomains. Specifically, let the tensor A in (1) be written as $A(x) = \alpha(x)\hat{A}(x)$, where α is a positive function that is piecewise constant with constant value α_i on Ω_i . The uniform ellipticity then implies that there exists positive constants \hat{c}_*, \hat{c}^* such that

$$\hat{c}_* \alpha_i \xi^T \xi \leq \xi^T A(x) \xi \leq \hat{c}^* \alpha_i \xi^T \xi \quad \forall \xi \in \mathbb{R}^n, \forall x \in \Omega_i. \quad (25)$$

The bound on the condition number of the operator that arises in balancing domain decomposition will depend on \hat{c}_* and \hat{c}^* but will be independent of α_i and c_* and c^* in (2).

Following Mandel’s original exposition in [3], we now recall the balancing preconditioner in terms of matrices. A equivalent variational presentation is given in [6]. By an abuse of notation, we use the same symbol to denote an element in $\mathcal{N}(\Gamma)$ and its associated vector of values attained at the nodal degrees of freedom.

The balancing preconditioner is parameterized by two sets of matrices, a set of weighting matrices $\{W_i\}_{i=1}^M$ and a set of kernel generators $\{Z_i\}_{i=1}^M$. The weighting matrices $W_i : \mathcal{N}(\partial\Omega_i) \rightarrow \mathcal{N}(\partial\Omega_i)$ are chosen such that they form a decomposition of unity on $\mathcal{N}(\Gamma)$, i.e.

$$\sum_{i=1}^M N_i W_i N_i^T p = p \quad \forall p \in \mathcal{N}(\Gamma),$$

where N_i denotes the canonical inclusion mapping $N_i : \mathcal{N}(\partial\Omega_i) \rightarrow \mathcal{N}(\Gamma)$ by extending elements of $\mathcal{N}(\partial\Omega_i)$ by zero at all other degrees of freedom. A prescription for the weighting matrices that guarantees a convergence bound independent of coefficient jumps between subdomains is given in Lemma 12 below. For each subdomain Ω_i , let $n_i = \dim(\mathcal{N}(\partial\Omega_i))$, and select an $n_i \times m_i$ matrix Z_i of full column rank with $0 \leq m_i \leq n_i$, such that

$$\text{Ker } S_i \subset \text{Range } Z_i, \quad i = 1, \dots, M. \quad (26)$$

For the scalar, second order, elliptic problems we consider in this paper, $\text{Ker } S_i$ is empty if there is Dirichlet data imposed on any part of $\partial\Omega_i \cap \partial\Omega$, otherwise it is the set of functions that have the same value at all the nodes on $\partial\Omega_i$. From the kernel generators, we construct a “coarse space”, $\mathcal{N}_H \subset \mathcal{N}(\Gamma)$, defined by

$$\mathcal{N}_H = \{p \in \mathcal{N}(\Gamma) : p = \sum_{i=1}^M N_i W_i z, z \in \text{Range } Z_i\}.$$

We say that $q \in \mathcal{N}(\Gamma)$ is *balanced* if it is orthogonal to \mathcal{N}_H ; that is,

$$Z_i^T W_i^T N_i^T q = 0, \quad i = 1, \dots, M. \quad (27)$$

The process of replacing r by a balanced $q = r - Sw$, $w \in \mathcal{N}_H$, will be called *balancing* and involves solving a “coarse grid problem” over the space \mathcal{N}_H described in (28) and (29) below.

The action of the balancing preconditioner M_{bal} is defined by the following algorithm.

Algorithm 1 Given $r \in \mathcal{N}(\Gamma)$, compute $M_{\text{bal}}^{-1}r$ as follows. Balance the original residual by solving the auxiliary problem for unknown vectors $\lambda_j \in \mathbb{R}^{m_j}$,

$$Z_i^T W_i^T N_i^T (r - S \sum_{j=1}^M N_j W_j Z_j \lambda_j) = 0, \quad i = 1, \dots, M \quad (28)$$

and set

$$q = r - S \sum_{j=1}^M N_j W_j Z_j \lambda_j, \quad q_i = W_i^T N_i^T q, \quad i = 1, \dots, M. \quad (29)$$

Find any solution $u_i \in \mathcal{N}(\partial\Omega_i)$ for each of the local problems

$$S_i u_i = q_i, \quad i = 1, \dots, M, \quad (30)$$

balance the residual by solving the auxiliary problem for $\mu_j \in \mathbb{R}^{m_j}$,

$$Z_i^T W_i^T N_i^T (r - S \sum_{j=1}^M N_j W_j (u_j + Z_j \mu_j)) = 0, \quad i = 1, \dots, M, \quad (31)$$

and set

$$M_{\text{bal}}^{-1}r = \sum_{i=1}^M N_i W_i (u_i + Z_i \mu_i). \quad (32)$$

If some $m_j = 0$, then Z_j as well as the block unknowns μ_j and λ_j are void and the j -th block equation is taken out of (28) and (31).

In [3], it was proven that Algorithm 1 implements a well defined operator that is symmetric and positive definite. An abstract bound on the condition number of $M_{\text{bal}}^{-1}S$ was also given. We will use the following lemma proven in [5] to determine a bound on the condition number of the preconditioned system for the application to the nonconforming discretization.

Lemma 12 For subdomain Ω_i , define the weighting map W_i as multiplication of the nodal values such that for any $p_i \in \mathcal{N}(\partial\Omega_i)$ and each nodal point $\nu \in \partial\Omega_i$,

$$(W_i p_i)(\nu) = \frac{\alpha_i}{\sum_{\{j: m_i \in \partial\Omega_j\}} \alpha_j} p_i(\nu). \quad (33)$$

Assume that there exists a number R so that

$$\frac{1}{\alpha_j} s_j(N_j^T N_i p_i, N_j^T N_i p_i) \leq \frac{1}{\alpha_i} R s_i(p_i, p_i) \quad (34)$$

for all $i, j = 1, \dots, M$ and all $p_i \in \mathcal{N}(\partial\Omega_i)$ that are orthogonal to the range of Z_i . Then there exists a constant C not dependent on h , H or R , so that the condition number $\kappa(M_{\text{bal}}^{-1}S)$ of the preconditioned system satisfies

$$\kappa(M_{\text{bal}}^{-1}S) \leq C R.$$

Since most nonconforming methods, like the mixed finite elements considered in [6], do not have nodal degrees of freedom at vertices in two dimensions and vertices and edges in three dimensions, the analysis of [6] is directly applicable to the nonconforming case. However, in keeping with the philosophy of this paper, we will allow vertex and edge degrees of freedom and will deduce the general nonconforming theory from the conforming theory. The following definitions and lemma from [5] will provide the essential conforming theory that we need for the nonconforming case.

Definition 1 Any vertex, edge, and, in the 3D case, face, of the interfaces between subdomains $\{\Omega_i\}$ will be called a glob. A glob is understood to be relatively open; for example, an edge does not contain its endpoints. We will also identify a glob with the set of the degrees of freedom associated with it. The set of all globs will be denoted by \mathcal{G} .

Definition 2 For a glob G , define the selection operator $E_G : U_h(\Gamma) \rightarrow U_h(\Gamma)$ as follows: for $\phi \in U_h(\Gamma)$, $E_G\phi$ is the unique function in $U_h(\Gamma)$ that has the same values as ϕ on the degrees of freedom in G , and all other degrees of freedom of $E_G\phi$ are zero.

Note that the union of all globs disjointly cover the set of all degrees of freedom of $U_h(\Gamma)$, and the mappings E_G are projections that induce a decomposition of unity on $U_h(\Gamma)$, $\sum_{G \in \mathcal{G}} E_G = I$.

Lemma 13 There exists a constant C such that for all $p_i \in \mathcal{N}(\partial\Omega_i)$ that are orthogonal to the range of Z_i and for all globs $G \in \partial\Omega_i \cap \partial\Omega_j$

$$|E_G \tilde{\mathcal{I}}^{\partial\Omega_i} p_i|_{1/2, \partial\Omega_j}^2 \leq C(1 + \log(H/h))^2 |\tilde{\mathcal{I}}^{\partial\Omega_i} \phi|_{1/2, \partial\Omega_i}^2.$$

Proof. The proposition follows from Lemmas 3.7, 4.6 and 5.1 of [5]. \square

We now prove a bound on the condition number for the preconditioned system in balancing domain decomposition for nonconforming elements.

Theorem 14 The interface operator S preconditioned by the balancing preconditioner M_{bal} defined in Algorithm 1 with weighting maps defined in (33) has a condition number $\kappa(M_{\text{bal}}^{-1}S)$ satisfying the bound

$$\kappa(M_{\text{bal}}^{-1}S) \leq C(1 + \log(H/h))^2 \quad (35)$$

in both two and three dimensions with the constant C independent of h , H , and α_i .

Proof. By tracing back the dependence on the coefficients, it is easy to prove the following refinement of Theorem 8:

$$s_i(p, p) \simeq \alpha_i |\tilde{\mathcal{I}}^{\partial\Omega_i} p|_{1/2, \partial\Omega_i}^2 \quad \forall p \in \mathcal{N}(\partial\Omega_i), \quad (36)$$

with equivalence constants that no longer depending on c_* and c^* in (2), but only on \hat{c}_* and \hat{c}^* in (25).

For $p_i \in \mathcal{N}(\partial\Omega_i)$, we may decompose $N_j^T N_i p_i$ as the nodal interpolant of globs:

$$N_j^T N_i p_i = \left(\sum_{G \subset \partial\Omega_j} \mathcal{I}_G^{\mathcal{N}} (E_G \tilde{\mathcal{I}}^{\partial\Omega_i} p_i) \right)_{|\partial\Omega_j}.$$

Let $|p_i|_{S_i}^2 = s_i(p_i, p_i)$. Considering those p_i that are orthogonal to the range of Z_i , working one glob at a time, and using (36), (19), Lemma 13 and (36) in that order, we have

$$\begin{aligned} |\mathcal{I}_F^N(E_G \tilde{\mathcal{I}}^{\partial\Omega_i} p_i)|_{S_j}^2 &\leq \alpha_j C |\tilde{\mathcal{I}}^{\partial\Omega_i} E_G \tilde{\mathcal{I}}^{\partial\Omega_i} p_i|_{1/2, \partial\Omega_j}^2 \leq \alpha_j C |E_G \tilde{\mathcal{I}}^{\partial\Omega_i} p_i|_{1/2, \partial\Omega_j}^2 \\ &\leq \alpha_j C (1 + \log(H/h))^2 |\tilde{\mathcal{I}}^{\partial\Omega_i} p_i|_{1/2, \partial\Omega_i}^2 \\ &\leq \frac{\alpha_j}{\alpha_i} C (1 + \log(H/h))^2 |p_i|_{S_i}^2. \end{aligned} \quad (37)$$

By the construction of the decomposition, there is an a priori maximum number of globs that intersect $\partial\Omega_i \cap \partial\Omega_j$. Summing over such globs, we conclude that

$$s_j(N_j^T N_i p_i, N_j^T N_i p_i) \leq \frac{\alpha_j}{\alpha_i} C (1 + \log(H/h))^2 s_i(p_i, p_i).$$

The proof is completed by appealing to the bound in Lemma 12. \square

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